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Invariance Analysis and Closed-form Solutions for The Beam Equation in Timoshenko Model

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Abstract

Our research focuses on a fourth-order partial differential equation (PDE) that arises from the Timoshenko model for beams. This PDE pertains to situations where the elastic moduli remain constant and an external load, represented as F, is applied. We thoroughly analyze Lie symmetries and categorize the various types of applied forces. Initially, the principal Lie algebra is two-dimensional, but in certain noteworthy cases, it extends to three dimensions or even more. For each specific case, we derive the optimal system, which serves as a foundation for symmetry reductions, transforming the original PDE into ordinary differential equations. In certain instances, we successfully identify exact solutions using this reduction process. Additionally, we delve into the conservation laws using a direct method proposed by Anco, with a particular focus on specific classes within the equation. The findings we have presented in our study are indeed original and innovative. This study serves as compelling evidence for the robustness and efficacy of the Lie symmetry method, showcasing its ability to provide valuable insights and solutions in the realm of mathematical analysis.

Keywords: Timoshenko beam equation; fourth-order partial differential equation; symmetry classification; exact solution; reductions.

1 Introduction

The beams are structures commonly used in buildings, bridges, and mechanical constructions. The simple Euler-Bernoulli theory [5, 4] is based upon considering only transverse bending. While this model is suitable for slender beams, it does not adequately represent many practical situations. Timoshenko in 1922 proposed a model which considers transverse displacement as well as the rotation effects. The governing equations in the presence of applied force are a system of partial differential equations (PDEs) [7],

$$\begin{aligned} \rho A\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(\kappa AG(\frac{\partial v(x,t)}{\partial x} - \varphi(x,t))\right) &= F(v), \\ \rho I\frac{\partial^2 \varphi(x,t)}{\partial t^2} - \frac{\partial}{\partial x}(EI\frac{\partial \varphi}{\partial x}) - \kappa AG\left(\frac{\partial v(x,t)}{\partial x} - \varphi(x,t)\right) &= 0, \end{aligned} \tag{1}$$

where $v = v(x,t)$ is the deflection or the transverse displacement of the beam, $\varphi = \varphi(x,t)$ the angular displacement, A cross-section area, ρ the density of the beam, G shear modulus, κ Timoshenko shear coefficient, I the area moment of the cross-section and $F = F(v)$ the distributed load.

Following [7], the system (1) can be merged into one-fourth order PDE by assuming that the rotation is mainly caused by bending and the effect of distortion due to shear can be ignored. Based on the above we can merge the system of PDEs (1) in one equation [8],

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} - \rho I[1+\gamma]\frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} + \frac{\rho^2 I\gamma}{E}\frac{\partial^4 v(x,t)}{\partial t^4} = F, \tag{2}$$

where γ is a parameter dependent on the shape of the cross-sectional area and E is the elastic modulus.

Dividing equation (2) by EI, and introducing the notations $\lambda_1 = \frac{\rho A}{EI}$, $\lambda_2 = \frac{\rho I[1+\gamma]}{EI}$, $\lambda_3 = \frac{\rho^2\gamma}{E^2}$, we are left with the equation,

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \lambda_1\frac{\partial^2 v(x,t)}{\partial t^2} - \lambda_2\frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} + \lambda_3\frac{\partial^4 v(x,t)}{\partial t^4} = F(v). \tag{3}$$

In equation (3), if $\lambda_i = 0$, for $i = 1, 2, 3$, we are left with static Euler-Bernoulli beam equation, which is fully studied in [4] and Fatima et al. [6] from symmetry point of view. If $\lambda_i = 0$, for $i = 2, 3$, then equation (3) is reduced to the dynamic Euler-Bernoulli beam equation, which is also studied from the Lie symmetry point of view by Bokhari et al. [5], and Soh [12]. If $\lambda_3 = 0$ in equation (3) we are left with the beam equation in the Rayleigh model [14], which is thought to be an improvement over the dynamic Euler-Bernoulli beam theory since it takes into account the rotational inertia of the beam's cross-section [7]. If $\lambda_2 = 0$ we get the Shear beam model, this model adds the effect of shear distortion without rotational inertia to the Euler-Bernoulli model [14]. If none of the λ_i coefficients are zero, we have the Timoshenko model, which is the focus of our paper and represents the most comprehensive form of equation (3). Interestingly, this equation has not been explored by previous authors in terms of exact solutions. In our study, we employ the potent Lie symmetry method, a robust technique for obtaining exact solutions, to analyze this equation. So, in this study, we consider the following PDE;

$$\frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} + \frac{\partial^4 v(x,t)}{\partial t^4} = F(v), \tag{4}$$

where the applied load F is a general function of v and ρ, A, I are constants.

The Lie symmetry approach [10, 15] is a powerful and well-documented method for dealing with exact solutions of differential equations, notably non-linear PDEs. Due to this importance in the final section, we investigate the final form (4) from the symmetry point of view and formulate its conservation laws for some classes by the direct method. The fourth-order beam equation, like its second-order wave equation counterpart, it turns out, displays interesting conservation properties, most of which are tied in with the symmetry structure of the equation. These differ for a zero applied force F, the linear Klein-Gordon type $F = \delta v$, and the case $F = \delta v^n$. In literature, various strategies exist for handling non-linear PDEs [9, 13].

2 Lie Classification

Following is a brief summary of the Lie symmetry [11, 1], the vector field for Eq. (2) is

$$\aleph = \xi(x,t,v)\frac{\partial}{\partial x} + \tau(x,t,v)\frac{\partial}{\partial t} + \eta(x,t,v)\frac{\partial}{\partial v}\cdot \tag{5}$$

The fourth order prolonged generator required for Eq. (4) is given by

$$\aleph^{[4]} = \aleph + \sum_{s=1}^{4} \zeta_{i_1\cdots i_s}\frac{\partial}{\partial v_{i_1\cdots i_s}}\cdot \tag{6}$$

Then by invariance condition of Eq. (4) we get

$$\aleph^{[4]}(v_{xxxx} + v_{tt} - v_{xxtt} + v_{tttt} - F(v))|_{\text{Eq}(4)} = 0. \tag{7}$$

Separating different polynomials in v and its derivatives condition (7) gives rise to an overdetermined system of PDEs. We solve this system for the resulting determining equations,

$$\xi = c_1, \qquad \tau = c_2, \qquad \eta = c_3 v + \gamma, \tag{8}$$

with γ satisfying the classifying equation

$$-(c_3 v + \gamma)F' + c_3 F + \gamma_{xxxx} + \gamma_{tt} - \gamma_{xxtt} + \gamma_{tttt} = 0. \tag{9}$$

In the equations (8) and (9), c_1, c_2 and c_3 are constants, $\gamma = \gamma(x,t)$ and F is the applied load dependent on v. If F is arbitrary non-linear function in v, then we get directly from Equations (8) and (9) the infinitesimals

$$\xi = c_1, \qquad \tau = c_2, \qquad \eta = 0. \tag{10}$$

This implies for arbitrary $F(v)$, Eq. (4) has the following Lie symmetry generators

$$\aleph_1 = \frac{\partial}{\partial x}, \qquad \aleph_2 = \frac{\partial}{\partial t}, \tag{11}$$

which constitutes a two-dimensional symmetry algebra. It is also known as the principal algebra of Eq. (4).

The following manifestations for the given load function F are determined by differentiating the classifying relation (9) concerning v.

$$\begin{array}{lll} i. & F(v) = \alpha v + \beta, & \alpha \neq 0, \\ ii. & F(v) = c_1(\alpha v + \beta)^{\sigma} + c_2, & \alpha \neq 0, \quad \sigma \neq 0, 1, \\ iii. & F(v) = c_1 e^{\alpha v} + c_2, & \alpha \neq 0, \\ iv. & F(v) = c_1 \ln(\alpha v + \beta) + c_2, & \alpha \neq 0, \end{array} \tag{12}$$

where α , β , c_1 , c_2 and σ are constants. But unfortunately, no one of the cases for non-linear $F(v)$ will extend the principal algebra, So we are left with the linear case

$$F = \alpha v + \beta. \tag{13}$$

The principal algebra in this case extends by one in addition to the infinite superposition generator. The Lie algebra is spanned by (11) and

$$\begin{aligned} \aleph_3 &= \left(v + \frac{\beta}{\alpha}\right)\frac{\partial}{\partial v}, \\ \aleph_\omega &= \omega\frac{\partial}{\partial v}, \end{aligned} \tag{14}$$

with ω satisfy the linear PDE $\omega_{xxxx} + \omega_{tt} - \omega_{xxtt} + \omega_{tttt} = 0$. As an example, we will consider the polynomial solution of the infinite part. Suppose the polynomial is of the form

$$\omega(x,t) = c_4x^3t + c_5x^3 + c_6x^2t + c_7x^2 + c_8xt + c_9x + c_{10}t + c_{11}, \tag{15}$$

which satisfies $\omega_{xxxx} + \omega_{tt} - \omega_{xxtt} + \omega_{tttt} = 0$. Then, the two-dimensional principal algebra will extend to the eleven-dimensional Lie algebra spanned by the minimal algebra (11) and $\aleph_3$ above in addition to

$$\begin{aligned} &\aleph_4 = x^3t\frac{\partial}{\partial v}, \quad \aleph_5 = x^3\frac{\partial}{\partial v}, \quad \aleph_6 = x^2t\frac{\partial}{\partial v}, \quad \aleph_7 = x^2\frac{\partial}{\partial v}, \\ &\aleph_8 = xt\frac{\partial}{\partial v}, \quad \aleph_9 = x\frac{\partial}{\partial v}, \quad \aleph_{10} = t\frac{\partial}{\partial v}, \quad \aleph_{11} = \frac{\partial}{\partial v}. \end{aligned} \tag{16}$$

3 Lie Symmetries and Optimal System

In this section, we focus on performing a symmetry classification of the beam Eq. (4) while considering various possible load functions.

(1) When $\mathbf{F(v)}$ is arbitrary.

$$\aleph_1 = \frac{\partial}{\partial t}, \quad \aleph_2 = \frac{\partial}{\partial x}. \tag{17}$$

Case A: $\mathbf{F(v)} = \alpha\mathbf{v} + \beta$.

$$\begin{aligned} &\aleph_1 = \frac{\partial}{\partial t}, \quad \aleph_2 = \frac{\partial}{\partial v}, \quad \aleph_3 = \frac{\partial}{\partial x}, \quad \aleph_4 = t\frac{\partial}{\partial v}, \\ &\aleph_5 = x\frac{\partial}{\partial v}, \quad \aleph_6 = x^2\frac{\partial}{\partial v}, \quad \aleph_7 = x^3\frac{\partial}{\partial v}, \quad \aleph_8 = \left(v + \frac{\beta}{\alpha}\right)\frac{\partial}{\partial v}, \\ &\aleph_9 = xt\frac{\partial}{\partial v}, \quad \aleph_{10} = x^2t\frac{\partial}{\partial v}, \quad \aleph_{11} = x^3t\frac{\partial}{\partial v}. \end{aligned} \tag{18}$$

Case B: $\mathbf{F(v)} = \beta$.

$$\begin{aligned} &\aleph_1 = \frac{\partial}{\partial t}, \quad \aleph_2 = \frac{\partial}{\partial v}, \quad \aleph_3 = \frac{\partial}{\partial x}, \\ &\aleph_4 = t\frac{\partial}{\partial v}, \quad \aleph_5 = x\frac{\partial}{\partial v}, \quad \aleph_6 = xt\frac{\partial}{\partial v}. \end{aligned} \tag{19}$$

Case C: $\mathbf{F}(\mathbf{v}) = \lambda_\mathbf{1}(\alpha\mathbf{v}+\beta)^\sigma + \lambda_\mathbf{2}$.

$$\aleph_1 = \frac{\partial}{\partial t}, \qquad \aleph_2 = \frac{\partial}{\partial x}. \tag{20}$$

Case D: $\mathbf{F}(\mathbf{v}) = \lambda_\mathbf{1}\mathbf{e}^{\alpha\mathbf{v}} + \lambda_\mathbf{2}$.

$$\aleph_1 = \frac{\partial}{\partial t}, \qquad \aleph_2 = \frac{\partial}{\partial x}. \tag{21}$$

Case E: $\mathbf{F}(\mathbf{v}) = \lambda_\mathbf{1}\ln(\alpha\mathbf{v}+\beta) + \lambda_\mathbf{2}$.

$$\aleph_1 = \frac{\partial}{\partial t}, \qquad \aleph_2 = \frac{\partial}{\partial x}. \tag{22}$$

3.1 Optimal system

(1) For arbitrary $\mathbf{F}(\mathbf{v})$.
The commutation relation is satisfied by the basis elements,

$$[\aleph_m, \aleph_n] = 0, \qquad m, n = 1, 2. \tag{23}$$

One can write the adjoint action representation as,

$$Ad\big(\exp(\epsilon\aleph_m).\aleph_n\big) = \aleph_n - \epsilon[\aleph_m,\aleph_n] + \frac{\epsilon^2}{2!}\big[\aleph_m,[\aleph_m,\aleph_n]\big] - \dots. \tag{24}$$

We take into account a general element $\aleph$ of $\mathcal{L}^2$ given by,

$$\aleph = \theta_1\aleph_1 + \theta_2\aleph_2. \tag{25}$$

Because commutation relations are zero, the vector form cannot be reduced.

So, for $\theta_1 \neq 0$, $\theta_2 \neq 0$, we have $\aleph_1 + c\aleph_2$, $c \neq 0$. If $\theta_1 = 0$, then we have $\aleph_2$. If $\theta_2 = 0$, then we have $\aleph_1$.

Hence, the one-dimensional optimal system of (17) is given by,

$$\big\{\aleph_1, \aleph_2, \aleph_1 + c\aleph_2\big\}. \tag{26}$$

Since symmetry generators for **Case C**, **Case D** and **Case E** are the same as in the arbitrary case, therefore, Eq. (26) represents an optimal system of (20), (21) and (22).

Case A: $\mathbf{F}(\mathbf{v}) = \alpha\mathbf{v} + \beta$.

Table 1: Commutator Table.

$[\aleph_i,\aleph_j]$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$	$\aleph_7$	$\aleph_8$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_1$	0	0	0	$\aleph_2$	0	0	0	0	$\aleph_5$	$\aleph_6$	$\aleph_7$
$\aleph_2$	0	0	0	0	0	0	0	$\aleph_2$	0	0	0
$\aleph_3$	0	0	0	0	$\aleph_2$	$2\aleph_5$	$3\aleph_6$	0	$\aleph_4$	$2\aleph_9$	$3\aleph_{10}$
$\aleph_4$	-$\aleph_2$	0	0	0	0	0	0	$\aleph_4$	0	0	0
$\aleph_5$	0	0	-$\aleph_2$	0	0	0	0	$\aleph_5$	0	0	0
$\aleph_6$	0	0	-$2\aleph_5$	0	0	0	0	$\aleph_6$	0	0	0
$\aleph_7$	0	0	-$3\aleph_6$	0	0	0	0	$\aleph_7$	0	0	0
$\aleph_8$	0	$-\aleph_2$	0	-$\aleph_4$	-$\aleph_5$	-$\aleph_6$	-$\aleph_7$	0	-$\aleph_9$	-$\aleph_{10}$	-$\aleph_{11}$
$\aleph_9$	-$\aleph_5$	0	-$\aleph_4$	0	0	0	0	$\aleph_9$	0	0	0
$\aleph_{10}$	-$\aleph_6$	0	-$2\aleph_9$	0	0	0	0	$\aleph_{10}$	0	0	0
$\aleph_{11}$	-$\aleph_7$	0	-$3\aleph_{10}$	0	0	0	0	$\aleph_{11}$	0	0	0

Table 2: Adjoint Table.

$Ad(e^\epsilon)$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_1$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4 - \epsilon\aleph_2$	$\aleph_5$	$\aleph_6$
$\aleph_2$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_3$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5 - \epsilon\aleph_2$	$\aleph_6 + \epsilon^2\aleph_2 - 2\epsilon\aleph_5$
$\aleph_4$	$\aleph_1 + \epsilon\aleph_2$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_5$	$\aleph_1$	$\aleph_2$	$\aleph_3 + \epsilon\aleph_2$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_6$	$\aleph_1$	$\aleph_2$	$\aleph_3 + 2\epsilon\aleph_5$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_7$	$\aleph_1$	$\aleph_2$	$\aleph_3 + 3\epsilon\aleph_6$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_8$	$\aleph_1$	$e^\epsilon\aleph_2$	$\aleph_3$	$e^\epsilon\aleph_4$	$e^\epsilon\aleph_5$	$e^\epsilon\aleph_6$
$\aleph_9$	$\aleph_1 + \epsilon\aleph_5$	$\aleph_2$	$\aleph_3 + \epsilon\aleph_4$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_{10}$	$\aleph_1 + \epsilon\aleph_6$	$\aleph_2$	$\aleph_3 + 2\epsilon\aleph_9$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_{11}$	$\aleph_1 + \epsilon\aleph_7$	$\aleph_2$	$\aleph_3 + 3\epsilon\aleph_{10}$	$\aleph_4$	$\aleph_5$	$\aleph_6$

Table 3: Adjoint Table.

$Ad(e^\epsilon)$	$\aleph_7$	$\aleph_8$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_1$	$\aleph_7$	$\aleph_8$	$\aleph_9 - \epsilon\aleph_5$	$\aleph_{10} - \epsilon\aleph_6$	$\aleph_{11} - \epsilon\aleph_7$
$\aleph_2$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_2$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_3$	$\aleph_7 - \epsilon^3\aleph_2 + 3\epsilon^2\aleph_5 - 3\epsilon\aleph_6$	$\aleph_8$	$\aleph_9 - \epsilon\aleph_4$	$\aleph_{10} + \epsilon^2\aleph_4 - 2\epsilon\aleph_9$	$\aleph_{11} - \epsilon^3\aleph_4 + 3\epsilon^2\aleph_9 - 3\epsilon\aleph_{10}$
$\aleph_4$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_4$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_5$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_5$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_6$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_6$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_7$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_7$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_8$	$e^\epsilon\aleph_7$	$\aleph_8$	$e^\epsilon\aleph_9$	$e^\epsilon\aleph_{10}$	$e^\epsilon\aleph_{11}$
$\aleph_9$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_9$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_{10}$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_{10}$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$
$\aleph_{11}$	$\aleph_7$	$\aleph_8 - \epsilon\aleph_{11}$	$\aleph_9$	$\aleph_{10}$	$\aleph_{11}$

Consider a general element $\aleph \in \mathcal{L}^{11}$ given by,

$$\aleph = \theta_1\aleph_1 + \theta_2\aleph_2 + \theta_3\aleph_3 + \theta_4\aleph_4 + \theta_5\aleph_5 + \theta_6\aleph_6 + \theta_7\aleph_7 + \theta_8\aleph_8 + \theta_9\aleph_9 + \theta_{10}\aleph_{10} + \theta_{11}\aleph_{11}. \tag{27}$$

Case 1: $\theta_1 \neq 0, \theta_3 = \theta_4 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0.$
By adjoint actions of $\aleph_4, \aleph_9, \aleph_{10}$ and $\aleph_{11}$ we get $\aleph = \aleph_1$.

Case 2: $\theta_1 \neq 0, \theta_3 \neq 0, \theta_4 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0.$
By adjoint actions of $\aleph_4, \aleph_9, \aleph_{10}$ and $\aleph_{11}$ we get $\aleph = \aleph_1 + c\aleph_3, c \neq 0$.

Case 3: $\theta_1 \neq 0, \theta_3 = 0, \theta_4 \neq 0, \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0.$
By adjoint actions of $\aleph_4, \aleph_9, \aleph_{10}, \aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_1 \pm \aleph_4$.

Case 4: $\theta_1 \neq 0, \theta_8 \neq 0, \theta_3 = \theta_4 = \theta_9 = \theta_{10} = \theta_{11} = 0.$
By adjoint actions of $\aleph_4, \aleph_9, \aleph_{10}$ and $\aleph_{11}$ we get $\aleph = \aleph_1 + c\aleph_8, c \neq 0$.

Case 5: $\theta_1 \neq 0, \theta_9 \neq 0, \theta_3 = \theta_4 = \theta_8 = \theta_{10} = \theta_{11} = 0.$
By adjoint actions of $\aleph_4, \aleph_9, \aleph_{10}, \aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_1 \pm \aleph_9$.

Case 6: $\theta_1 \neq 0, \theta_{10} \neq 0, \theta_3 = \theta_4 = \theta_8 = \theta_9 = \theta_{11} = 0$.
By adjoint actions of $\aleph_4$, $\aleph_9$, $\aleph_{10}$, $\aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_1 \pm \aleph_{10}$.

Case 7: $\theta_1 \neq 0, \theta_{11} \neq 0, \theta_3 = \theta_4 = \theta_8 = \theta_9 = \theta_{10} = 0$.
By adjoint actions of $\aleph_4$, $\aleph_9$, $\aleph_{10}$, $\aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_1 \pm \aleph_{11}$.

Case 8: $\theta_2 \neq 0, \theta_3 = \theta_1 = \theta_4 = \theta_5 = \theta_6 = \theta_7 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
We have $\aleph = \aleph_2$.

Case 9: $\theta_2 \neq 0, \theta_9 \neq 0, \theta_{11} = \theta_{10} = \theta_8 = \theta_7 = \theta_6 = \theta_5 = \theta_4 = \theta_3 = \theta_1 = 0$.
By adjoint action of $\aleph_8$ we get $\aleph = \aleph_2 + c\aleph_9$, $c \neq 0$.

Case 10: $\theta_2 \neq 0, \theta_1 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = \theta_7 = \theta_8 = \theta_9 = \theta_{11} = 0, \theta_{10} \neq 0$.
By adjoint action of $\aleph_8$ we get $\aleph = \aleph_2 + c\aleph_{10}$, $c \neq 0$.

Case 11: $\theta_{11} \neq 0, \theta_{10} = \theta_9 = \theta_8 = \theta_7 = \theta_6 = \theta_5 = \theta_4 = \theta_3 = 0, \theta_2 \neq 0, \theta_1 = 0$
By adjoint action of $\aleph_8$ we get $\aleph = \aleph_2 + c\aleph_{11}$, $c \neq 0$.

Case 12: $\theta_3 \neq 0, \theta_1 = \theta_7 = \theta_8 = \theta_{11} = 0$.
By adjoint actions of $\aleph_5$, $\aleph_6$, $\aleph_7$, $\aleph_9$, $\aleph_{10}$ and $\aleph_{11}$ we get $\aleph = \aleph_3$.

Case 13: $\theta_3 \neq 0, \theta_7 \neq 0, \theta_1 = \theta_8 = \theta_{11} = 0$.
By adjoint actions of $\aleph_5$, $\aleph_6$, $\aleph_7$, $\aleph_9$, $\aleph_{10}$, $\aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_3 \pm \aleph_7$.

Case 14: $\theta_3 \neq 0, \theta_8 \neq 0, \theta_1 = \theta_7 =, \theta_{11} = 0$.
By adjoint actions of $\aleph_5$, $\aleph_6$, $\aleph_7$, $\aleph_9$, $\aleph_{10}$ and $\aleph_{11}$ we get $\aleph = \aleph_3 + c\aleph_8$, $c \neq 0$.

Case 15: $\theta_3 \neq 0, \theta_1 = \theta_7 = \theta_8 = 0, \theta_{11} \neq 0$.
By adjoint actions of $\aleph_5$, $\aleph_6$, $\aleph_7$, $\aleph_9$, $\aleph_{10}$, $\aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_3 \pm \aleph_{11}$.

Case 16: $\theta_4 \neq 0, \theta_1 = \theta_3 = \theta_5 = \theta_6 = \theta_7 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_1$ we get $\aleph = \aleph_4$.

Case 17: $\theta_1 = \theta_3 = \theta_4 \neq 0, \theta_5 \neq 0, \theta_6 = \theta_7 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_1$ we get $\aleph = \aleph_4 + c\aleph_5$, $c \neq 0$.

Case 18: $\theta_1 = \theta_3 = 0, \theta_4 \neq 0, \theta_6 \neq 0, \theta_7 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_4 + c\aleph_6$, $c \neq 0$.

Case 19: $\theta_1 = \theta_3 = 0, \theta_4 \neq 0, \theta_7 \neq 0, \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_4 + c\aleph_7$, $c \neq 0$.

Case 20: $\theta_1 = \theta_3 = \theta_4 = \theta_5 \neq 0, \theta_6 = \theta_7 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_5$.

Case 21: $\theta_1 = \theta_3 = \theta_5 \neq 0, \theta_6 = \theta_7 = \theta_8 = \theta_{10} \neq 0, \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_5 + c\aleph_{10}$, $c \neq 0$.

Case 22: $\theta_1 = \theta_3 = 0, \theta_5 \neq 0, \theta_6 = \theta_7 = \theta_8 = 0, \theta_{11} \neq 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_5 + c\aleph_{11}$, $c \neq 0$.

Case 23: $\theta_1 = \theta_3 = \theta_4 = 0, \theta_6 \neq 0, \theta_7 = \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_6$.

Case 24: $\theta_1 = \theta_3 = 0, \theta_6 \neq 0, \theta_7 = \theta_8 = 0, \theta_9 \neq 0, \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_6 + c\aleph_9$, $c \neq 0$.

Case 25: $\theta_1 = \theta_3 = 0, \theta_6 \neq 0, \theta_7 = \theta_8 = 0, \theta_{11} \neq 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_6 + c\aleph_{11}$, $c \neq 0$.

Case 26: $\theta_1 = \theta_3 = \theta_4 = 0, \theta_7 \neq 0, \theta_8 = \theta_9 = \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_7$.

Case 27: $\theta_1 = \theta_3 = 0, \theta_7 \neq 0, \theta_8 = 0, \theta_9 \neq 0, \theta_{10} = \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_7 + c\aleph_9$, $c \neq 0$.

Case 28: $\theta_1 = \theta_3 = 0, \theta_7 \neq 0, \theta_8 = 0, \theta_{10} \neq 0, \theta_{11} = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_7 + c\aleph_{10}$, $c \neq 0$.

Case 29: $\theta_1 = 0, \theta_3 = 0, \theta_8 \neq 0$.
By adjoint actions of $\aleph_2, \aleph_4, \aleph_5, \aleph_6, \aleph_7, \aleph_9, \aleph_{10}, \aleph_{11}$ we get $\aleph = \aleph_8$.

Case 30: $\theta_1 = \theta_2 = \theta_3 = \theta_6 = \theta_7 = \theta_8 = 0, \theta_9 \neq 0, \theta_{10} = \theta_{11} = 0$.
By adjoint actions of $\aleph_1$ and $\aleph_3$ we get $\aleph = \aleph_9$.

Case 31: $\theta_1 = \theta_3 = \theta_7 = \theta_8 = 0, \theta_{10} \neq 0, \theta_{11} = 0$.
By adjoint actions of $\aleph_1$ and $\aleph_3$ we get $\aleph = \aleph_{10}$.

Case 32: $\theta_1 = 0, \theta_3 = 0, \theta_8 = 0, \theta_{11} \neq 0$.
By adjoint actions of $\aleph_1$ and $\aleph_3$ we get $\aleph = \aleph_{11}$.

Case 33: $\theta_1 \neq 0, \theta_3 \neq 0, \theta_8 \neq 0$.
By adjoint actions of $\aleph_4, \aleph_6, \aleph_7, \aleph_{10}$ and $\aleph_{11}$ we get $\aleph = \aleph_1 + c\aleph_3 + d\aleph_8$.

Case 34: $\theta_1 \neq 0, \theta_3 \neq 0, \theta_8 = 0, \theta_{11} \neq 0$.
By adjoint action of $\aleph_3, \aleph_{11}$ and $\aleph_8$ we get $\aleph = \aleph_1 + c\aleph_3 \pm \aleph_{11}$.

Case B: $\mathbf{F}(\mathbf{v}) = \beta$.

Table 4: Commutator Table.

$[\aleph_i, \aleph_j]$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_1$	0	0	0	$\aleph_2$	0	$\aleph_5$
$\aleph_2$	0	0	0	0	0	0
$\aleph_3$	0	0	0	0	$\aleph_2$	$\aleph_4$
$\aleph_4$	$-\aleph_2$	0	0	0	0	0
$\aleph_5$	0	0	$-\aleph_2$	0	0	0
$\aleph_6$	$-\aleph_5$	0	$-\aleph_4$	0	0	0

Table 5: Adjoint Table.

$[\aleph_i, \aleph_j]$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_1$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4 - \epsilon\aleph_2$	$\aleph_5$	$\aleph_6 - \epsilon\aleph_5$
$\aleph_2$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_3$	$\aleph_1$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5 - \epsilon\aleph_2$	$\aleph_6 - \epsilon\aleph_4$
$\aleph_4$	$\aleph_1 + \epsilon\aleph_2$	$\aleph_2$	$\aleph_3$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_5$	$\aleph_1$	$\aleph_2$	$\aleph_3 + \epsilon\aleph_2$	$\aleph_4$	$\aleph_5$	$\aleph_6$
$\aleph_6$	$\aleph_1 + \epsilon\aleph_5$	$\aleph_2$	$\aleph_3 + \epsilon\aleph_4$	$\aleph_4$	$\aleph_5$	$\aleph_6$

Consider a general element $\aleph \in \mathcal{L}^6$ given by,

$$\aleph = \theta_1\aleph_1 + \theta_2\aleph_2 + \theta_3\aleph_3 + \theta_4\aleph_4 + \theta_5\aleph_5 + \theta_6\aleph_6. \tag{28}$$

Case 1: $\theta_1 \neq 0, \theta_3 = 0, \theta_4 = 0, \theta_6 = 0$.
By adjoint actions of $\aleph_4$ and $\aleph_6$ we get $\aleph = \aleph_1$.

Case 2: $\theta_1 \neq 0, \theta_3 \neq 0, \theta_6 = 0$.
By adjoint actions of $\aleph_1$ and $\aleph_6$ we get $\aleph = \aleph_1 + c\aleph_3$, $c \neq 0$.

Case 3: $\theta_1 \neq 0, \theta_3 = 0, \theta_4 \neq 0, \theta_6 = 0$.
By adjoint action of $\aleph_1$ we get $\aleph = \aleph_1 + c\aleph_4$, $c \neq 0$.

Case 4: $\theta_1 \neq 0, \theta_3 = 0, \theta_6 \neq 0$.
By adjoint actions of $\aleph_1$ and $\aleph_3$ we get $\aleph = \aleph_1 + c\aleph_6$, $c \neq 0$.

Case 5: $\theta_2 \neq 0, \theta_1 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$.
We get $\aleph = \aleph_2$.

Case 6: $\theta_2 \neq 0, \theta_1 = \theta_3 = \theta_4 = \theta_5 = 0, \theta_6 \neq 0$.
We get $\aleph = \aleph_2 + c\aleph_6$, $c \neq 0$.

Case 7: $\theta_3 \neq 0, \theta_1 = \theta_5 = \theta_6 = 0$.
By adjoint actions of $\aleph_5$ and $\aleph_6$ we get $\aleph = \aleph_3$.

Case 8: $\theta_3 \neq 0, \theta_5 \neq 0, \theta_1 = \theta_6 = 0$.
By adjoint actions of $\aleph_5$ and $\aleph_6$ we get $\aleph = \aleph_3 + c\aleph_5$, $c \neq 0$.

Case 9: $\theta_1 = 0, \theta_3 \neq 0, \theta_6 \neq 0$.
By adjoint actions of $\aleph_3$ and $\aleph_1$ we get $\aleph = \aleph_3 + c\aleph_6$, $c \neq 0$.

Case 10: $\theta_1 = 0, \theta_3 = 0, \theta_4 \neq 0, \theta_5 = 0, \theta_6 = 0$.
By adjoint action of $\aleph_1$ we get $\aleph = \aleph_4$.

Case 11: $\theta_1 = 0, \theta_3 = 0, \theta_4 \neq 0, \theta_5 \neq 0, \theta_6 = 0$.
By adjoint action of $\aleph_1$ we get $\aleph = \aleph_4 + c\aleph_5$, $c \neq 0$.

Case 12: $\theta_1 = 0, \theta_3 = 0, \theta_4 = 0, \theta_5 \neq 0, \theta_6 = 0$.
By adjoint action of $\aleph_3$ we get $\aleph = \aleph_5$.

Case 13: $\theta_1 = 0, \theta_3 = 0, \theta_6 \neq 0$.
By adjoint actions of $\aleph_1$ and $\aleph_3$ we get $\aleph = \aleph_6$.

Case 14: $\theta_1 \neq 0, \theta_3 \neq 0, \theta_6 \neq 0$.
By adjoint actions of $\aleph_1$ and $\aleph_3$ we get $\aleph = \aleph_1 + c\aleph_3 + d\aleph_6$, $c, d \neq 0$.

4 Similarity Reductions and Invariant Solutions

(1) When $\mathbf{F}(\mathbf{v})$ is arbitrary.

Symmetry reduction by $\aleph_1$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{0}, \tag{29}$$

which gives $v = g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - F\big(g(r)\big) = 0. \tag{30}$$

The function F determines the precise answer to the equation (4).

Symmetry reduction by $\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{0}, \tag{31}$$

giving $v = g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - F(g(r)) = 0. \tag{32}$$

The function F determines the precise answer to equation (32).

Symmetry reduction by $\aleph_1 + c\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{0}, \tag{33}$$

which gives $v = g(r)$, $r = t - \frac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1)g'''' + c^4 g'' - c^4 F(g(r)) = 0. \tag{34}$$

The function F determines the precise answer to equation (34).

Case A: $\mathbf{F}(\mathbf{v}) = \alpha\mathbf{v} + \beta$.

Symmetry reduction by $\aleph_1$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{0}, \tag{35}$$

giving $v = g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - \alpha g - \beta = 0, \tag{36}$$

this gives,

$$g(r) = c_1 e^{\alpha^{\frac{1}{4}} r} + c_2 e^{-\alpha^{\frac{1}{4}} r} + c_3 e^{-i\alpha^{\frac{1}{4}} r} + c_4 e^{i\alpha^{\frac{1}{4}} r} - \frac{\beta}{\alpha}. \tag{37}$$

So, the solution in the original variables becomes,

$$v(x,t) = c_1 e^{\alpha^{\frac{1}{4}} x} + c_2 e^{-\alpha^{\frac{1}{4}} x} + c_3 e^{-i\alpha^{\frac{1}{4}} x} + c_4 e^{i\alpha^{\frac{1}{4}} x} - \frac{\beta}{\alpha}. \tag{38}$$

Symmetry reduction by $\aleph_3$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{0}, \tag{39}$$

which gives $v = g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \alpha g - \beta = 0, \tag{40}$$

this gives,

$$\begin{aligned} g(r) = {} & c_1 e^{-\frac{\sqrt{-2-2\sqrt{1+4\alpha}}r}{2}} + c_2 e^{\frac{\sqrt{-2-2\sqrt{1+4\alpha}}r}{2}} + c_3 e^{-\frac{\sqrt{-2+2\sqrt{1+4\alpha}}r}{2}} \\ & + c_4 e^{\frac{\sqrt{-2+2\sqrt{1+4\alpha}}r}{2}} - \frac{\beta}{\alpha}. \end{aligned} \tag{41}$$

So, the solution in the original variables becomes,

$$\begin{aligned} v(x,t) = {} & c_1 e^{-\frac{\sqrt{-2-2\sqrt{1+4\alpha}}t}{2}} + c_2 e^{\frac{\sqrt{-2-2\sqrt{1+4\alpha}}t}{2}} + c_3 e^{-\frac{\sqrt{-2+2\sqrt{1+4\alpha}}t}{2}} \\ & + c_4 e^{\frac{\sqrt{-2+2\sqrt{1+4\alpha}}t}{2}} - \frac{\beta}{\alpha}. \end{aligned} \tag{42}$$

Symmetry reduction by $\aleph_1 + c\aleph_3$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{0}, \tag{43}$$

giving $v = g(r)$, $r = t - \frac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1)g'''' - c^4(\alpha g - g'' + \beta) = 0. \tag{44}$$

So, the solution in the original variables becomes,

$$\begin{aligned} v(x,t) = {} & c_1 e^{\frac{-\sqrt{-c^2+\sqrt{(4c^4-4c^2+4)\alpha+c^4}}(tc-x)}{\sqrt{2(c^4-c^2+1)}}} + c_2 e^{\frac{\sqrt{-c^2+\sqrt{(4c^4-4c^2+4)\alpha+c^4}}(tc-x)}{\sqrt{2(c^4-c^2+1)}}} \\ & + c_3 e^{\frac{-i\sqrt{c^2+\sqrt{(4c^4-4c^2+4)\alpha+c^4}}(tc-x)}{\sqrt{2(c^4-c^2+1)}}} + c_4 e^{\frac{i\sqrt{c^2+\sqrt{(4c^4-4c^2+4)\alpha+c^4}}(tc-x)}{\sqrt{2(c^4-c^2+1)}}} - \frac{\beta}{\alpha}. \end{aligned} \tag{45}$$

Symmetry reduction by $\aleph_1 + c\aleph_8$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{v + \frac{\beta}{\alpha}}, \tag{46}$$

giving $v = e^{tc}g(r) - \dfrac{\beta}{\alpha}$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - c^2 g'' + c^4 g + c^2 g - \alpha g = 0. \tag{47}$$

So, the solution in the original variables becomes,

$$\begin{aligned} v(x,t) = {} & c_1 e^{tc - \frac{\sqrt{2c^2 - 2\sqrt{-3c^4 - 4c^2 + 4\alpha}}x}{2}} + c_2 e^{tc + \frac{\sqrt{2c^2 - 2\sqrt{-3c^4 - 4c^2 + 4\alpha}}x}{2}} \\ & + c_3 e^{tc - \frac{\sqrt{2c^2 + 2\sqrt{-3c^4 - 4c^2 + 4\alpha}}x}{2}} + c_4 e^{tc + \frac{\sqrt{2c^2 + 2\sqrt{-3c^4 - 4c^2 + 4\alpha}}x}{2}} - \frac{\beta}{\alpha}. \end{aligned} \tag{48}$$

Symmetry reduction by $\aleph_3 + c\aleph_8$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{v + \frac{\beta}{\alpha}}, \tag{49}$$

which gives $v = e^{cx}g(r) - \dfrac{\beta}{\alpha}$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' - c^2 g'' + g'' + c^4 g - \alpha g = 0. \tag{50}$$

So, the solution in the original variables becomes,

$$\begin{aligned} v(x,t) = {} & c_1 e^{cx - \frac{\sqrt{-2 + 2c^2 - 2\sqrt{-3c^4 - 2c^2 + 4\alpha + 1}t}}{2}} + c_2 e^{cx + \frac{\sqrt{-2 + 2c^2 - 2\sqrt{-3c^4 - 2c^2 + 4\alpha + 1}t}}{2}} \\ & + c_3 e^{cx - \frac{\sqrt{-2 + 2c^2 + 2\sqrt{-3c^4 - 2c^2 + 4\alpha + 1}t}}{2}} + c_4 e^{cx + \frac{\sqrt{-2 + 2c^2 + 2\sqrt{-3c^4 - 2c^2 + 4\alpha + 1}t}}{2}} - \frac{\beta}{\alpha}. \end{aligned} \tag{51}$$

Case B: $\mathbf{F(v)} = \beta$.

Symmetry reduction by $\aleph_1$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{0}, \tag{52}$$

giving $v = g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - \beta = 0, \tag{53}$$

this gives,

$$g(r) = \frac{1}{24}\beta r^4 + \frac{c_1}{6} r^3 + \frac{c_2}{2} r^2 + c_3 r + c_4. \tag{54}$$

So, the solution in the original variables becomes,

$$v(x,t) = \frac{1}{24}\beta x^4 + \frac{c_1}{6} x^3 + \frac{c_2}{2} x^2 + c_3 x + c_4. \tag{55}$$

Symmetry reduction by $\aleph_3$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{0}, \tag{56}$$

which gives $v = g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \beta = 0, \tag{57}$$

this gives,

$$g(r) = \frac{\beta}{2} r^2 - c_1 \cos r - c_2 \sin r + c_3 r + c_4. \tag{58}$$

So, the solution in the original variables becomes,

$$v(x,t) = \frac{\beta}{2} t^2 - c_1 \cos t - c_2 \sin t + c_3 t + c_4. \tag{59}$$

Symmetry reduction by $\aleph_1 + c\aleph_3$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{0}, \tag{60}$$

which gives $v = g(r)$, $r = t - \dfrac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1) g'''' - c^4 (\beta - g'') = 0. \tag{61}$$

So, the solution in the original variables becomes,

$$\begin{aligned} v(x,t) = \frac{1}{2c^4} \Bigg(& -2c_1 (c^4 - c^2 + 1) \cos\left(\frac{c(tc - x)}{\sqrt{c^4 - c^2 + 1}} \right) \\ & - 2c_2 (c^4 - c^2 + 1) \sin\left(\frac{c(tc - x)}{\sqrt{c^4 - c^2 + 1}} \right) \\ & + \Big((\beta t^2 + 2c_3 t + 2c_4) c^2 - 2x(\beta t + c_3) c + \beta x^2 \Big) c^2 \Bigg). \end{aligned} \tag{62}$$

Symmetry reduction by $\aleph_1 + c\aleph_4$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{ct}, \tag{63}$$

giving $v = \dfrac{t^2 c}{2} + g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' + c - \beta = 0. \tag{64}$$

So, the solution in the original variables becomes,

$$v(x,t) = \frac{t^2 c}{2} + \frac{x^4}{24} (\beta - c) + \frac{c_1}{6} x^3 + \frac{c_2}{2} x^2 + c_3 x + c_4. \tag{65}$$

Symmetry reduction by $\aleph_1 + c\aleph_6$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{cxt}, \tag{66}$$

which gives $v = \dfrac{t^2cx}{2} + g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' + cg - \beta = 0. \tag{67}$$

So, the solution in the original variables becomes,

$$v(x,t) = \frac{t^2cx}{2} - \frac{c}{120}x^5 + \frac{\beta}{24}x^4 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4. \tag{68}$$

Symmetry reduction by $\aleph_3 + c\aleph_5$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{cx}, \tag{69}$$

giving $v = \dfrac{cx^2}{2} + g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \beta = 0. \tag{70}$$

So, the solution in the original variables becomes,

$$v(x,t) = \frac{cx^2}{2} + \frac{\beta t^2}{2} - c_1\cos(t) - c_2\sin(t) + c_3t + c_4. \tag{71}$$

Symmetry reduction by $\aleph_1 + c\aleph_3 + d\aleph_6$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{xtd}, \tag{72}$$

giving $v = \dfrac{3cdtx^2 - x^3d}{6c^2} + g(r)$, $r = t - \dfrac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1)g'''' - c^4(\beta - g'') = 0. \tag{73}$$

So, the solution in the original variables becomes,

$$\begin{aligned} v(x,t) = \frac{1}{6c^4}\Bigg(& -6c_1(c^4 - c^2 + 1)\cos\left(\frac{c(tc - x)}{\sqrt{c^4 - c^2 + 1}}\right) \\ & - 6c_2(c^4 - c^2 + 1)\sin\left(\frac{c(tc - x)}{\sqrt{c^4 - c^2 + 1}}\right) \\ & + 3c^2((\beta t^2 + 2c_3t + 2c_4)c^2 + x(xtd - 2\beta t - 2c_3)c - \frac{d}{3}x^3 + \beta x^2)\Bigg). \end{aligned} \tag{74}$$

Symmetry reduction by $\aleph_3 + c\aleph_6$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{cxt}, \tag{75}$$

which gives $v = \dfrac{ctx^2}{2} + g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \beta = 0. \tag{76}$$

So, the solution in the original variables becomes,

$$v(x,t) = \frac{\beta t^2}{2} - c_1 \cos(t) - c_2 \sin(t) + \frac{(cx^2 + 2c_3)t}{2} + c_4. \tag{77}$$

Case C: $\mathbf{F(v) = \lambda_1(\alpha v + \beta)^\sigma + \lambda_2}$.

Symmetry reduction by $\aleph_1$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{0}, \tag{78}$$

giving $v = g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - \alpha g - \lambda_1(\alpha g + \beta)^\sigma - \lambda_2 = 0. \tag{79}$$

Symmetry reduction by $\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{0}, \tag{80}$$

which gives $v = g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \alpha g - \lambda_1(\alpha g + \beta)^\sigma - \lambda_2 = 0. \tag{81}$$

Symmetry reduction by $\aleph_1 + c\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{0}, \tag{82}$$

which gives $v = g(r)$, $r = t - \dfrac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1)g'''' + c^4(-\alpha g + g'' - \lambda_2) - c^4\lambda_1(\alpha g + \beta)^\sigma = 0. \tag{83}$$

Case D: $\mathbf{F(v) = \lambda_1 e^{\alpha v} + \lambda_2}$.

Symmetry reduction by $\aleph_1$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{0}, \tag{84}$$

giving $v = g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - \alpha g - \lambda_1 e^{\alpha g} - \lambda_2 = 0. \tag{85}$$

Symmetry reduction by $\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{0}, \tag{86}$$

which gives $v = g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \alpha g - \lambda_1 e^{\alpha g} - \lambda_2 = 0. \tag{87}$$

Symmetry reduction by $\aleph_1 + c\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{0}, \tag{88}$$

which gives $v = g(r)$, $r = t - \frac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1)g'''' - c^4(\lambda_1 e^{\alpha g} + \alpha g + \lambda_2 - g'') = 0. \tag{89}$$

Case E: $\mathbf{F}(\mathbf{v}) = \lambda_1 \ln(\alpha \mathbf{v} + \beta) + \lambda_2$.

Symmetry reduction by $\aleph_1$:
The associated characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dv}{0}, \tag{90}$$

giving $v = g(r)$, $r = x$. This transformation reduces the Eq. (4) to

$$g'''' - \alpha g - \lambda_1 \ln(\alpha g + \beta) - \lambda_2 = 0. \tag{91}$$

Symmetry reduction by $\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{dv}{0}, \tag{92}$$

which gives $v = g(r)$, $r = t$. This transformation reduces the Eq. (4) to

$$g'''' + g'' - \alpha g - \lambda_1 \ln(\alpha g + \beta) - \lambda_2 = 0. \tag{93}$$

Symmetry reduction by $\aleph_1 + c\aleph_2$:
The associated characteristic equation is,

$$\frac{dx}{c} = \frac{dt}{1} = \frac{dv}{0}, \tag{94}$$

which gives $v = g(r)$, $r = t - \frac{x}{c}$. This transformation reduces the Eq. (4) to

$$(c^4 - c^2 + 1)g'''' - c^4(\lambda_1 \ln(\alpha g + \beta) + \alpha g + \lambda_2 - g'') = 0. \tag{95}$$

5 Conservation Laws

The fourth-order beam equation, like its second-order wave equation counterpart as it turns out, displays interesting conservation properties, most of which are tied in with the symmetry structure of the equation. For a zero applied force F and linear Klein-Gordon type $F = \delta v$, the conservation laws are infinite; most of which are consequences of higher-order 'multipliers'. Briefly, the multiplier approach requires the determination of a multiplier,Q, say, for which $Q\Big[v_{tttt} + v_{tt} - v_{ttxx} + v_{xxxx} - F(v)\Big]$ is a total divergence [2, 3]. We list the conserved flows below. The conserved flow (T^x, T^t) is obtained by applying the conservation law $D_tTt + D_xTx = 0$ to the solutions of the differential equation. If $D_tT^t + D_xT^x = 0$ identically, the vector (T^x, T^t) is called a 'trivial' conserved vector.

The list below enumerates the multipliers Q and corresponding nontrivial flows (T^x, T^t), where appropriate, the form $v^{(i,j)}$ represents $v_{ix\,jt}$.

a. $F = 0$,

i.

$$x : \left(\frac{1}{6}[-3xv^{(1,2)} + 6xv^{(3,0)} + v^{(0,2)} - 6v^{(2,0)}], xv^{(0,1)} + xv^{(0,3)} - \frac{1}{2}xv^{(2,1)} + \frac{1}{3}v^{(1,1)}\right).$$

ii.

$$t : \left(-\frac{1}{2}tv^{(1,2)} + tv^{(3,0)} + \frac{1}{3}v^{(1,1)}, \frac{1}{6}\Big[6tv^{(0,1)} + 6tv^{(0,3)} - 3tv^{(2,1)} - 6v^{(0,2)} + v^{(2,0)} - 6v\Big]\right).$$

iii.

$$xt : \left(\frac{1}{6}\left(-3txv^{(1,2)} + 6txv^{(3,0)} + tv^{(0,2)} - 6tv^{(2,0)} + 2xv^{(1,1)} - 2v^{(0,1)}\right),\right.$$
$$\left.\frac{1}{6}\left(6txv^{(0,1)} + 6txv^{(0,3)} - 3txv^{(2,1)} + 2tv^{(1,1)} - 6xv^{(0,2)} + xv^{(2,0)} - 2v^{(1,0)} - 6xv\right)\right).$$

iv.

$$1 : \left(v^{(3,0)} - \frac{1}{2}v^{(1,2)}, v^{(0,1)} + v^{(0,3)} - \frac{1}{2}v^{(2,1)}\right).$$

v.

$$v_t : \left(\frac{1}{12}\Big[3vv^{(1,3)} - 6vv^{(3,1)} - v^{(0,3)}v^{(1,0)} + 6v^{(2,1)}v^{(1,0)} + 3v^{(0,2)}v^{(1,1)}\right.$$
$$- 5v^{(0,1)}v^{(1,2)} - 6v^{(1,1)}v^{(2,0)} + 6v^{(0,1)}v^{(3,0)}\Big],$$
$$\frac{1}{12}\Big[- 3vv^{(2,2)} + 6vv^{(4,0)} + 6\left(v^{(0,1)}\right)^2 + 4\left(3v^{(0,3)} - v^{(2,1)}\right)v^{(0,1)}$$
$$\left. - 6\left(v^{(0,2)}\right)^2 + 2\left(v^{(1,1)}\right)^2 - 2v^{(1,0)}v^{(1,2)} + v^{(0,2)}v^{(2,0)}\Big]\right).$$

vi.

$$v_x : \left(\frac{1}{12} \Big[3vv^{(1,3)} - 6vv^{(3,1)} - v^{(0,3)}v^{(1,0)} + 6v^{(2,1)}v^{(1,0)} + 3v^{(0,2)}v^{(1,1)} - 5v^{(0,1)}v^{(1,2)} - 6v^{(1,1)}v^{(2,0)} + 6v^{(0,1)}v^{(3,0)} \Big], \frac{1}{12} \Big[-3vv^{(2,2)} + 6vv^{(4,0)} + 6\left(v^{(0,1)}\right)^2 + 4\left(3v^{(0,3)} - v^{(2,1)}\right)v^{(0,1)} - 6\left(v^{(0,2)}\right)^2 + 2\left(v^{(1,1)}\right)^2 - 2v^{(1,0)}v^{(1,2)} + v^{(0,2)}v^{(2,0)} \Big] \right).$$

vii.

$$\sin(t+x) : \left(\frac{1}{12} \Big[3vv^{(1,3)} - 6vv^{(3,1)} - v^{(0,3)}v^{(1,0)} + 6v^{(2,1)}v^{(1,0)} + 3v^{(0,2)}v^{(1,1)} - 5v^{(0,1)}v^{(1,2)} - 6v^{(1,1)}v^{(2,0)} + 6v^{(0,1)}v^{(3,0)} \Big], \frac{1}{12} \Big[-3vv^{(2,2)} + 6vv^{(4,0)} + 6\left(v^{(0,1)}\right)^2 + 4\left(3v^{(0,3)} - v^{(2,1)}\right)v^{(0,1)} - 6\left(v^{(0,2)}\right)^2 + 2\left(v^{(1,1)}\right)^2 - 2v^{(1,0)}v^{(1,2)} + v^{(0,2)}v^{(2,0)} \Big] \right).$$

b. Now consider the beam equation with linear force in v, viz., $v_{tttt} + v_{tt} - v_{ttxx} + v_{xxxx} - \delta v = 0$. We obtain the following conserved flows.

i.

$$e^{(-\delta)^{1/4}x} : \left(\frac{1}{6} e^{\sqrt[4]{-\delta}x} \Big[-6(-\delta)^{3/4}v + \sqrt[4]{-\delta}v_{tt} - 6\sqrt[4]{-\delta}v_{xx} + 6\sqrt{-\delta}v_x - 3v_{xtt} + 6v_{xxx} \Big], -\frac{1}{6} e^{\sqrt[4]{-\delta}x} \Big[-2\sqrt[4]{-\delta}v_{xt} + \left(\sqrt{-\delta} - 6\right)v_t + 3v_{xxt} - 6v_{ttt} \Big] \right).$$

ii.

$$e^{-(-\delta)^{1/4}x} : \left(\frac{1}{6} e^{\sqrt[4]{-\delta}(-x)} \Big[6(-\delta)^{3/4}v + \sqrt[4]{-\delta}\,(-v_{tt}) + 6\sqrt[4]{-\delta}v_{xx} + 6\sqrt{-\delta}v_x - 3v_{xtt} + 6v_{xxx} \Big], \frac{1}{6} e^{\sqrt[4]{-\delta}(-x)} \Big[-2\sqrt[4]{-\delta}v_{xt} - \left(\sqrt{-\delta} - 6\right)v_t - 3v_{xxt} + 6v_{ttt} \Big] \right).$$

iii.

$$e^{\frac{1}{2}\sqrt{-2(1+\sqrt{1-4\delta})}t}:\left(\frac{1}{12}e^{\frac{\sqrt{-\sqrt{1-4\delta}-1}t}{\sqrt{2}}}\left[2\left(\sqrt{2}\sqrt{-\sqrt{1-4\delta}-1}v_{xt}-3v_{xtt}+6v_{xxx}\right)\right.\right.$$
$$\left.+\left(\sqrt{1-4\delta}+1\right)v_x\right],$$
$$\frac{1}{12}e^{\frac{\sqrt{-\sqrt{1-4\delta}-1}t}{\sqrt{2}}}\left[3\sqrt{2}\sqrt{-\sqrt{1-4\delta}-1}\left(\sqrt{1-4\delta}-1\right)v\right.$$
$$-6\left(\sqrt{1-4\delta}-1\right)v_t-6\sqrt{2}\sqrt{-\sqrt{1-4\delta}-1}v_{tt}$$
$$\left.\left.+\sqrt{2}\sqrt{-\sqrt{1-4\delta}-1}v_{xx}-6v_{xxt}+12v_{ttt}\right]\right).$$

v.

$$v_t:\left(\frac{1}{12}\left[3vv_{xttt}-6vv_{xxxt}-v_{ttt}v_x+6v_xv_{xxt}+3v_{tt}v_{xt}-5v_tv_{xtt}-6v_{xx}v_{xt}+6v_tv_{xxx}\right],\right.$$
$$\frac{1}{12}\left[-3v\left(v_{xxtt}-2v_{xxxx}\right)+6\delta v^2+2{v_{xt}}^2-2v_xv_{xtt}+v_{tt}v_{xx}\right.$$
$$\left.\left.+4v_t\left(3v_{ttt}-v_{xxt}\right)+6{v_t}^2-6{v_{tt}}^2\right]\right).$$

vi.

$$v_x:\left(\frac{1}{12}\left[v\left(-3v_{xxtt}+6v_{tt}+6v_{tttt}\right)+6\delta v^2+2{v_{xt}}^2-4v_xv_{xtt}+v_{tt}v_{xx}-2v_tv_{xxt}\right.\right.$$
$$\left.-6{v_{xx}}^2+12v_xv_{xxx}\right],$$
$$\frac{1}{12}\left[-6vv_{xt}-6vv_{xttt}+3vv_{xxxt}+6v_{ttt}v_x-5v_xv_{xxt}-6v_{tt}v_{xt}+3v_{xx}v_{xt}\right.$$
$$\left.\left.+v_t\left(6v_{xtt}+6v_x-v_{xxx}\right)\right]\right).$$

vii.

$$\sin(t+x):\left(\frac{1}{12}\left[3vv^{(1,3)}-6vv^{(3,1)}-v^{(0,3)}v^{(1,0)}+6v^{(2,1)}v^{(1,0)}+3v^{(0,2)}v^{(1,1)}\right.\right.$$
$$\left.-5v^{(0,1)}v^{(1,2)}-6v^{(1,1)}v^{(2,0)}+6v^{(0,1)}v^{(3,0)}\right],$$
$$\frac{1}{12}\left[-3vv^{(2,2)}+6vv^{(4,0)}+6\left(v^{(0,1)}\right)^2+4\left(3v^{(0,3)}-v^{(2,1)}\right)v^{(0,1)}\right.$$
$$\left.\left.-6\left(v^{(0,2)}\right)^2+2\left(v^{(1,1)}\right)^2-2v^{(1,0)}v^{(1,2)}+v^{(0,2)}v^{(2,0)}\right]\right).$$

There exists another three, based on the multipliers $e^{-\frac{1}{2}\sqrt{-2(1+\sqrt{1-4\delta})}t}$, $e^{\frac{1}{2}\sqrt{-2(1-\sqrt{1-4\delta})}t}$ and $e^{-\frac{1}{2}\sqrt{-2(1-\sqrt{1-4\delta})}t}$.

Note:
In the above cases, **a.** and **b.**, there exist infinitely many conservation laws based on higher-order multipliers such as ∂_{xxx}, ∂_{xxt}, etc. For example, in **b.** we have

$$\begin{aligned} v_{xxx}: \Bigg(&\frac{1}{12}\Big[3v(x,t)\left(4\delta v_{xx}+2v_{xxtt}+2v_{xxtttt}-v_{xxxxtt}\right)-6\delta {v_x}^2+v_x\left(-6v_{xtt}-6v_{xtttt}+5v_{xxxtt}\right)\\ &+6v_{tt}v_{xx}+6v_{tttt}v_{xx}-6v_{xx}v_{xxtt}-3v_{xxx}v_{xtt}+2v_{xt}v_{xxxt}+v_{tt}v_{xxxx}-2v_t v_{xxxxt}\\ &+6{v_{xxx}}^2\Big],\frac{1}{12}\Big[-6v(x,t)v_{xxxt}-6v(x,t)v_{xxxttt}+3v(x,t)v_{xxxxxt}+6v_{ttt}v_{xxx}-3v_{xxx}v_{xxt}\\ &-6v_{tt}v_{xxxt}+v_{xx}v_{xxxt}+2v_{xxxx}v_{xt}-2v_x v_{xxxxt}+v_t\left(6v_{xxxtt}+6v_{xxx}-v_{xxxxx}\right)\Big]\Bigg). \end{aligned} \tag{96}$$

c. For a general polynomial non-linearity in v, we have $v_{tttt}+v_{tt}-v_{ttxx}+v_{xxxx}-\delta v^n=0$, for which we have the following multipliers and corresponding conserved flows.

i.

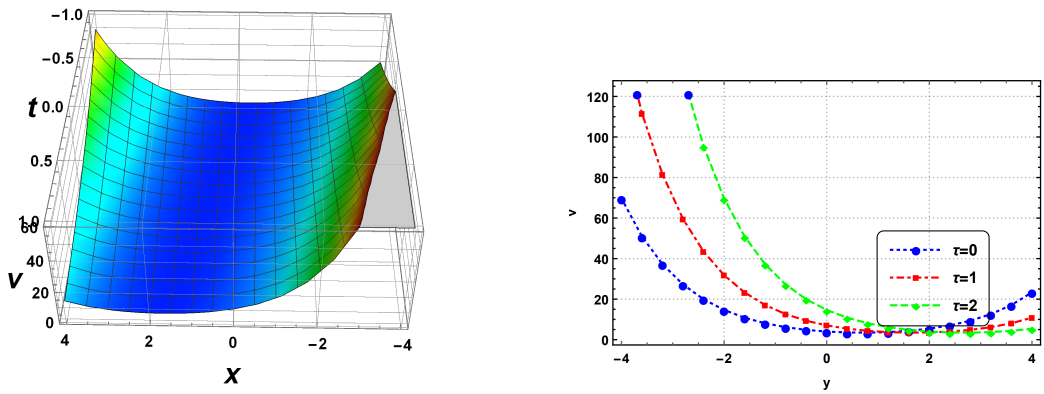
$$\begin{aligned} v_t: \Bigg(&\frac{1}{12}\Big[3vv_{xttt}-6vv_{xxxt}-v_{ttt}v_x+6v_xv_{xxt}+3v_{tt}v_{xt}-5v_tv_{xtt}-6v_{xx}v_{xt}+6v_tv_{xxx}\Big],\\ &\frac{1}{12}\Big[-3v\left(v_{xxtt}-2v_{xxxx}\right)+\frac{12\delta v^{n+1}}{n+1}+2{v_{xt}}^2-2v_xv_{xtt}+v_{tt}v_{xx}\\ &+4v_t\left(3v_{ttt}-v_{xxt}\right)+6{v_t}^2-6{v_{tt}}^2\Big]\Bigg). \end{aligned}$$

ii.

$$\begin{aligned} v_x: \Bigg(&\frac{1}{12}\Big[v\left(6\left(v_{tt}+v_{tttt}\right)-3v_{xxtt}\right)+\frac{12\delta v^{n+1}}{n+1}+2{v_{xt}}^2+v_{xx}\left(v_{tt}-6v_{xx}\right)-2v_tv_{xxt}\\ &-4v_x\left(v_{xtt}-3v_{xxx}\right)\Big],\\ &\frac{1}{12}\Big[-6vv_{xt}-6vv_{xttt}+3vv_{xxxt}+6v_{ttt}v_x-5v_xv_{xxt}-6v_{tt}v_{xt}\\ &+3v_{xx}v_{xt}+v_t\left(6v_{xtt}+6v_x-v_{xxx}\right)\Big]\Bigg). \end{aligned}$$

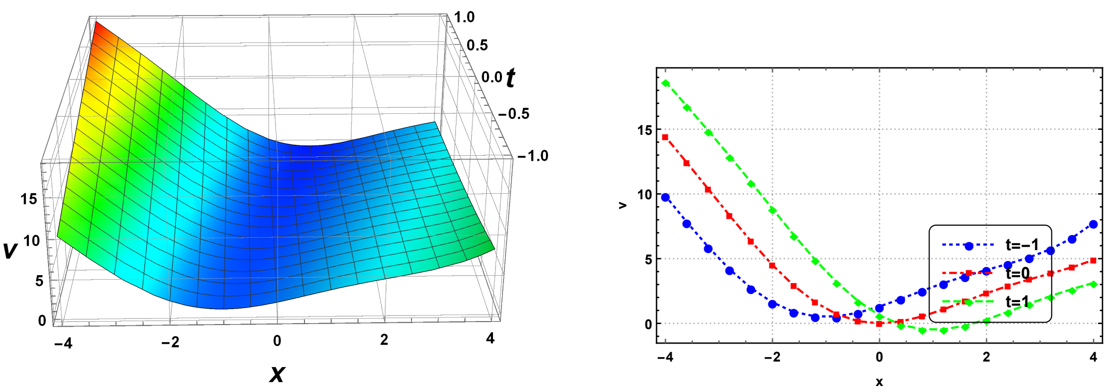
6 Physical Interpretation

In this section, our focus is directed towards the examination of the solutions derived through graphical analysis. It's well understood that this type of graphical analysis plays a pivotal role in comprehending the physical dynamics inherent to the model under consideration. In our current investigation, the graphical representation offers insights into the characteristics of the transverse displacement of the beam (v) within the model being studied. The visual representations, as depicted in Figures 1 through 4, distinctly portray the nature of the transverse displacement of the beam within the context of the Timoshenko beam model (4).



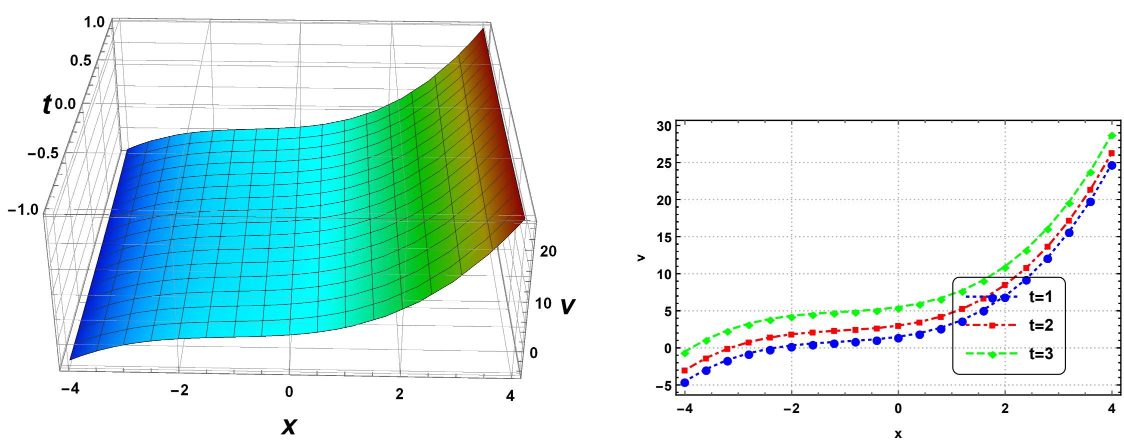
(a) 3D plot (b) 2D plot

Figure 1: Nature of the transverse displacement of the beam by solution (45) with all the parameters are set to 1.



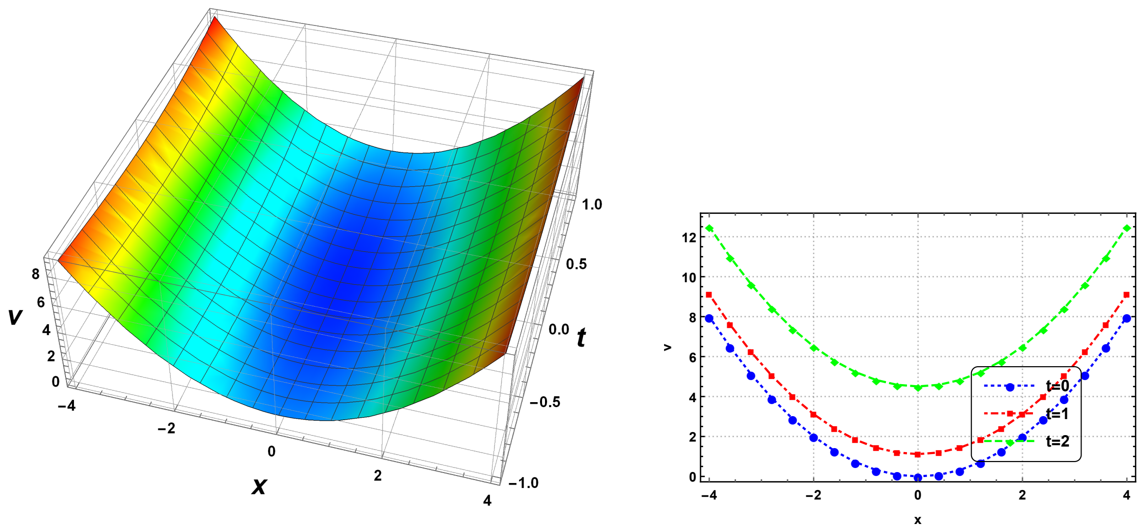
(a) 3D plot (b) 2D plot

Figure 2: Nature of the transverse displacement of the beam by solution (62) with $\beta = 2$ and all other parameters are set to 1.



(a) 3D plot (b) 2D plot

Figure 3: Nature of the transverse displacement of the beam by solution (65) with all the parameters are set to 1.



(a) 3D plot (b) 2D plot

Figure 4: Nature of the transverse displacement of the beam by solution (71) with all the parameters are set to 1.

7 Conclusion

We have performed a complete Lie symmetry classification of the fourth-order partial differential equation (4) arising from the Timoshenko beam model, with applied load F dependent on transverse displacement v. The results reported in the literature for the Euler Bernoulli beam were obtained by neglecting shear and rotational effects. The principal Lie algebra is found to be two-dimensional for the arbitrarily applied load. The algebra extends to an infinite-dimensional algebra for the constant and linear applied load. We systematically identified all conceivable invariant variables and their associated reductions for each vector field within a one-dimensional optimal system of subalgebras. These reductions resulted in ordinary differential equations, and we presented them comprehensively. These reductions were characterized as optimal, as they enabled us to derive all non-similar invariant solutions through symmetry transformations from the solutions of the reduced ODEs. We have also shown that the fourth-order beam equation displays interesting conservation properties. For a zero applied force F and linear Klein-Gordon type $F = \delta v$, the conservation laws are infinite; most of which are consequences of higher-order 'multipliers'. Our findings strongly validate the reliability and effectiveness of the Lie symmetry method in the context of beam theory. This success has inspired us to tackle more challenging nonlinear and complex problems within beam theory using the same method in future research endeavors.

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Conflicts of Interest The authors declare no conflict of interest.

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